FUNDAMENTAL SOLUTION OF AN UNSTEADY TRANSPORT EQUATION IN SOLID GEOMETRY

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We obtain a fundamental solution of an unsteady transport equation in solid geometry. This solution can, subsequently be used to obtain the field characteristics when the source function is not symmetric (solution of the problem when the source is arbitrary can be obtained by convolution of the source function with the given basic solution).

It should be noted that the considerable amount of attention given to the transport equation stems from its importance in the fields of the theoretical atmospheric- and hydro-optics and in the problems of neutron diffusion and gamma radiation; however, the overwhelming majority of investigators stop at the problems where the required field (or the particle distribution density) possesses a plane, cyllindrical or spherical symmetry (see e.g. [1 to 3]).

We shall investigate the stated problem using the terminology pertaining to the diffusion of particles in a scattering medium. The following notation shall be used: t is time; ω_0 $(|\omega_0| = 1)$ is the unit direction vector of the emitted particles; $\mathbf{x}$ and ω $(|\omega| = 1)$ are the spatial and angular arguments of the required particle distribution density $I(t, \mathbf{x}, \omega)$; v is the velocity of the particles; h is the scattering cross section; β is the total scattering cross section; $\gamma(\cos\alpha)$ is the scattering coefficient expressed in the terms of the cosine of the scattering angle α; Ω denotes the surface of a unit sphere; $\mathbf{x}\cdot\mathbf{y}$ is the scalar product of $\mathbf{x}$ and $\mathbf{y}$ and the step function is determined by the condition

$$\theta(x) = 1, \quad x > 0; \quad \theta(x) = 0, \quad x < 0$$

The following restrictions are made: we assume that all physical parameters (h, β and the scattering coefficients) are independent of the particle velocity, that the scattering medium is homogeneous, isotropic and infinite (absence of the boundaries), that the scattering coefficient $\gamma(\cos\alpha)$ can be expanded into a finite series in the Legendre polynomials

$$\gamma(\cos\alpha) = \sum_{l=0}^{N} b_l P_l(\cos\alpha), \qquad b_0 = 1 \tag{1}$$

Under these assumptions we find, that to obtain the basic solution of an unsteady transport equation in solid geometry, we must solve the following equation for the particle distribution density $I(t, x, \omega)$:

$$\frac{1}{v}\frac{\partial I}{\partial t} + \omega\cdot\operatorname{grad} I + \beta I = \frac{h}{4\pi}\int_\Omega \gamma(\omega\cdot\omega') I(t, \mathbf{x}, \omega')\, d\omega' + \\ + \frac{1}{v}\delta(t)\,\delta(\mathbf{x})\,\delta(\omega - \omega_0) \tag{2}$$

Taking into account the simple connection existing between the basic solution of our equation and the basic solution of the corresponding Cauchy's problem (basic solution of our equation corresponds to the basic solution of the Cauchy's problem for $t > 0$, see [4]), we can express the problem defined by (2) as a homogeneous transport equation

$$\frac{1}{v}\frac{\partial I}{\partial t} + \omega \cdot \operatorname{grad} I + \beta I = \frac{h}{4\pi}\int_{\Omega} \gamma(\omega\cdot\omega')\, I(t,\ \mathrm{x},\ \omega')\, d\omega' \qquad (3)$$

with initial conditions

$$I\,|_{t=0} = \delta(\mathrm{x})\,\delta(\omega - \omega_0) \qquad (4)$$

To solve (3) under initial condition (4) we shall expand $\delta(\mathrm{x})$ in [4] into plane waves, thus reducing the problem to one containing a single spatial dimension. We know [4], that in a three-dimensional space we have

$$\delta(\mathrm{x}) = -\frac{1}{8\pi^2}\int_{\Omega} \delta^{(2)}(\mathrm{x}\cdot\omega')\, d\omega' \qquad (5)$$

where $\delta^{(2)}$ denotes a second order derivative of a one-dimensional δ-function with respect to its argument.

Let us introduce a coordinate system with the origin at $\mathrm{x} = 0$ and the x- axis perpendicular to the plane $(\mathrm{x},\ \omega') = 0$ and let the direction of collimation ω in this coordinate system be defined by a polar angle θ counted from the positive direction of the x-axis and by the azimuthal angle φ in the plane $(\mathrm{x},\ \omega') = 0$.

In addition, let us put $\mu = \cos\theta$.

Taking into account the linearity of (3) and the relation (5), we shall first consider a formal plane source, whose particle distribution function $\psi(t,\ x,\ \omega)$ satisfies Eq.

$$\frac{1}{v}\frac{\partial \psi}{\partial t} + \mu\frac{\partial \psi}{\partial x} + \beta\psi = \frac{h}{4\pi}\int_{\Omega} \gamma(\omega\cdot\omega')\,\psi(t,\ x,\ \omega')\, d\omega' \qquad (6)$$

with the initial condition

$$\psi\,|_{t=0} = \delta^{(2)}(x)\delta(\omega - \omega_0) = \delta^{(2)}(x)\ \delta(\mu - \mu_0)\ \delta(\varphi - \varphi_0) \qquad (7)$$

Obviously, the required magnitude $I(t,\ \mathrm{x},\ \omega)$ is related to the function
$\psi(t,\ x,\ \omega) \equiv \psi$ thus $(t,\ x,\ \mu,\ \mu_0,\ \varphi,\ \varphi_0)$

$$I(t,\ \mathrm{x},\ \omega) = -\frac{1}{8\pi^2}\int_{\Omega} \psi(t,\ x = \mathrm{x}\cdot\omega',\ \mu = \omega\cdot\omega',\ \mu_0 = \omega_0\cdot\omega',\ \varphi,\ \varphi_0)\, d\omega' \qquad (8)$$

which, together with (5), describes the superposition of fields of the plane sources randomly orientated and emitting the particles in the direction ω_0.

In view of the singular character of the initial condition (7), we must seek the solution of (6) in the class of generalized functions. For notational brevity, we shall not distinguish between the generalized (continuous linear functionals in the space of infinitely differentiable finite functions) and the ordinary functions. Moreover, to simplify the notation further we shall, when referring to the arguments of the generalized functions, use the term 'point' with the relevant expression for the argument instead of the term 'neighborhood' requiring a more complicated expression for the argument.

To solve the Eq. (6) we shall expand the particle distribution density in the terms of the number of scattering collisions undergone by each particle. This method was given in [5] for the case of a plane, isotropic source. Thus we put (9)

$$\psi(t,\ x,\ \omega) = \sum_{n=0}^{\infty} \psi_n(t,\ x,\ \omega), \quad \psi_n(t,\ x,\ \omega) = \frac{e^{-\beta vt}}{(vt)^3}\frac{(hvt)^n}{n!}F_n(\eta,\ \omega), \quad \eta = \frac{x}{vt}$$

Here ψ_n denote the terms (in the physical sense) in the expression for the particle distribution density ψ, corresponding to the particles undergoing exactly n collisions over the period $[0, t]$.

Inserting (9) into (6) we obtain, for the unknown function F_n, the following set of recurrent relations

$$(\mu - \eta)\frac{\partial F_0}{\partial \eta} = 3F_0$$

$$(n-3)F_n + (\mu - \eta)\frac{\partial F_n}{\partial \eta} = \frac{n}{4\pi}\int_\Omega \gamma(\omega\cdot\omega')F_{n-1}(\eta, \omega')\,d\omega' \qquad (n \geqslant 1) \qquad (10)$$

Initial conditions satisfying the function F_n can be found by obtaining F_0 together with a recurrence relation between F_n and F_{n-1}. For this purpose we shall use the Green's function of (6), whose right hand side will be a known function of the source

$$G(t, t', x, x') = e^{-\beta v(t-t')}\,\theta(t-t')\,\delta(x - x' - v(t-t')\mu) \qquad (11)$$

Using the initial particle distribution as a source we obtain, from

$$\frac{1}{v}\frac{\partial\psi_0}{\partial t} + \mu\frac{\partial\psi_0}{\partial x} + \beta\psi_0 = \frac{1}{v}\,\delta^{(2)}(x)\,\delta(t)\,\delta(\mu-\mu_0)\,\delta(\varphi-\varphi_0)$$

the distribution density of the particles passed without collision

$$\psi_0 = \delta^{(2)}(x - \mu vt)\,\delta(\mu - \mu_0)\,\delta(\varphi - \varphi_0)\,e^{-\beta vt} \qquad (12)$$

which, together with (9), yields

$$F_0 = \delta^{(2)}(\mu - \eta)\,\delta(\mu - \mu_0)\,\delta(\varphi - \varphi_0) \qquad (13)$$

Assuming now that for $n \geq 1$ the source function has the form

$$f_n(t, x, \omega) = \frac{hv\theta(t)}{4\pi}\int_\Omega \gamma(\omega\cdot\omega')\,\psi_{n-1}(t, x, \omega')\,d\omega'$$

we find, using the Green's function (12), the relation between ψ_n and ψ_{n-1}

$$\psi_n(t, x, \omega) = \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dx' G(t, t', x, x')\, f_n(t', x'\, \omega) = \qquad (14)$$

$$= \frac{hv}{4\pi}e^{-\beta vt}\int_{-\infty}^{\infty}\theta(t')\,\theta(t-t')\,e^{\beta vt'}\,dt'\int_\Omega \gamma(\omega\cdot\omega')\,\psi_{n-1}(t', x - v(t-t')\mu, \omega')\,d\omega' \qquad (n \geqslant 1)$$

which, together with (9), yields

$$F_n(\eta, \omega) = \frac{n}{4\pi t^{n-3}}\int_{-\infty}^{\infty} t'^{n-4}\,\theta(t')\,\theta(t-t')\,dt'\int_\Omega \gamma(\omega\cdot\omega') \times F_{n-1} \times \left(\frac{x - v(t-t')\mu}{vt'}, \omega'\right) d\omega', \quad (n \geqslant 1) \qquad (15)$$

Formulas (12) and (14) will now show that, when $t \to 0$, $\psi_n \to 0$ if $n \geq 1$. Thus the quantity ψ_0 should satisfy the initial condition (7), in accordance with (9).

Let us now turn our attention to solving Eqs. (10). In view of the singularity of these equations, their solutions F_n should be obtained as linear combinations of the particular solutions $F_n^{\pm}$ which correspond to integrations around the singularity when $\eta = \mu$ in the upper and lower semiplane of the complex variable η respectively (see [5]). The first Eq. of (10) has two, linearly independent solutions

$$F_0^{\pm} = \lim_{\varepsilon \to +0} \frac{\delta(\mu - \mu_0)\,\delta(\varphi - \varphi_0)}{(\mu - \eta \mp i\varepsilon)^3} \tag{16}$$

which can be uniquely combined to yield

$$F_0 = \frac{1}{\pi i}(F_0^{+} - F_0^{-}) \tag{17}$$

satisfying the condition (13).

Let us find the form of the combination of $F_n^{\pm}$ defining F_n for $n \geq 1$. We shall consider the cases $\eta > \mu$ and $\eta < \mu$ separately*. Let us introduce the symbol $\sigma = \operatorname{sign}(\eta - \mu)$. Going, in (10), around the singularity, we obtain its two linearly independent solutions

$$F_n^{\pm}(\eta, \omega) = \frac{n}{4\pi} \lim_{\varepsilon \to +0} (\mu - \eta \mp i\varepsilon)^{n-3} \times$$
$$\times \left[\int_{a_\sigma}^{\eta} \frac{d\eta'}{(\mu - \eta' \mp i\varepsilon)^{n-2}} \int_{\Omega} \gamma(\omega \cdot \omega')\, F_{n-1}^{\pm}(\eta', \omega')\, d\omega' + c_n^{\pm}\right] \quad (n \geqslant 1) \tag{18}$$

where $c_n^{\pm}$ is a constant of integration and a_σ is the lower limit of integration chosen in some definite manner.

From (16) it follows that as $|\eta| \to \infty$, we have $F_0^{\pm} = 0(|\eta|^{-3})$ uniformly in $\omega \in \Omega$ and the function $F_0^{\pm}$ is analytic in η everywhere for $\eta \neq \mu$; thus when $n = 1$, we can put in (18) $a_\sigma = \sigma \cdot \infty$. Further, from (18) we find that when $\sigma\eta \to \infty$, we have $F_1^{\pm} = O(|\eta|^{-3})$; uniformly in $\omega \in \Omega$; therefore we can put $a\sigma = \sigma \cdot \infty$ for $n = 2$. Using now the formula (18) the required number of times, we find that we can assume $a_\sigma = \sigma \cdot \infty$, for any $n \geq 1$, since for $\sigma\eta \to \infty$, we have

$$F_n^{\pm} = O(|\eta|^{-3}) \tag{19}$$

uniformly in $\omega \in \Omega$.

Replacing now in (18) the variable of integration by

$$t' = t(\mu - \eta)/(\mu - \eta')$$

we obtain

$$F_n^{\pm}(\eta, \omega) = \frac{n}{4\pi t^{n-3}} \int_{-\infty}^{\infty} t'^{n-4}\theta(t')\,\theta(t - t')\,dt' \int_{\Omega} \gamma(\omega' \cdot \omega) \times \tag{20}$$
$$\times F_{n-1}^{\pm}\left(\frac{x - v(t - t')\mu}{vt'}, \omega'\right) d\omega' + \frac{n}{4\pi} c_n^{\pm} \lim_{\varepsilon \to +0} (\mu - \eta \mp i\varepsilon)^{n-3} \quad (n \geqslant 1)$$

Comparing (15) and (20) and taking into account (17) we find, that when $c_n^{\pm} = 0$, then the functions F_n for $(n \geq 1)$ can be expressed in terms of $F_n^{\pm}$ using the following relation analogous to (17):

$$F_n = \frac{1}{\pi i}(F_n^{+} - F_n^{-}) \tag{21}$$

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* The case $\eta = \mu$ is excluded, since the generalized function has no values at the separate points (see [4]).

In the following we shall use the relations (21), assuming that in (18) and (20) $c_n^{\pm}=0$.

Having found the required auxilliary relations, we shall return to our main problem i.e. to find $F_n^{\pm}$ from (10). Let us introduce the differential operator $D=\partial(\ldots)/\partial\eta$ defining its positive powers by $D^0=I$ (unit operator) and $D^k=\partial^k(\cdots)/\partial\eta^k$ $(k=1,2,\ldots)$. Moreover, we shall denote by D^{-1} and D^{-2} a single and double integration respectively, in the limits $\sigma\cdot\infty$ to η.

Going in (10) around the singularity and applying the operator D^{n-3} to both sides of this equation, we obtain

$$(\mu-\eta\mp i\varepsilon)\,D^{n-2}F_n^{\pm}=\frac{n}{4\pi}\int\limits_{\Omega}\gamma(\omega\cdot\omega')\,D^{n-3}F_{n-1}^{\pm}(\eta,\omega')\,d\omega' \qquad (n\geqslant 1) \tag{22}$$

Dividing now both parts of (22) by $(\mu-\eta\mp i\varepsilon)$ passing to the limit as $\epsilon\to+0$ and using the relation (1) together with the addition theorem for the Legendre polynomials, we obtain

$$D^{n-2}F_n^{\pm}=n\lim_{\varepsilon\to+0}\frac{1}{\mu-\eta\mp i\varepsilon}\sum_{l=0}^{N}\frac{b_l}{2l+1}\sum_{m=-l}^{l}Y_{lm}(\omega)\times$$
$$\times\int\limits_{\Omega}\overline{Y}_{lm}(\omega')\,D^{n-3}F_{n-1}^{\pm}(\eta,\omega')\,d\omega' \qquad (n\geqslant 1) \tag{23}$$

where the functions $\overline{Y}_{lm}(\omega)=Y_{lm}(\mu,\varphi)$ represent the spherical harmonics and $\overline{Y}_{lm}(\omega)$ their complex conjugate.

In the following we shall employ the vector and matrix notation; $||\mathbf{X}||_l$ will denote the l-th component of the vector $\mathbf{X}$, while $||A||_{ik}$ will denote an element of the matrix A. Subscripts l, i and k will start with the number $|m|\leq N$. We shall introduce $(N-|m|+1)$-component vectors $\Phi_{m,n}^{\pm}(\eta)$ and $\mathbf{Y}_m(\omega)$ and square $(N-|m|+1)$-th order matrices $R_m^{\pm}(\eta)$ and Bm:

$$\|\Phi_{m,n}^{\pm}(\eta)\|_l=\int\limits_{\Omega}\overline{Y}(\omega')\,D^{n-2}F_n^{\pm}(\eta,\omega')\,d\omega',\quad |m|\leqslant l\leqslant N,\quad n\geqslant 0 \tag{24}$$

$$\|\mathbf{Y}_m(\omega)\|_l=Y_{ml}(\omega),\qquad |m|\leqslant l\leqslant N \tag{25}$$

$$\|R_m^{\pm}(\eta)\|_{ik}=\lim_{\varepsilon\to+0}\int\limits_{\Omega}\frac{\overline{Y}_{im}(\omega')\,Y_{km}(\omega')\,d\omega'}{\mu'-\eta\mp i\varepsilon},\quad |m|\leqslant i,\ k\leqslant N \tag{26}$$

$$\|B_m\|_{ik}=\frac{b_k}{2k+1}\delta_{ik},\quad |m|\leqslant i,\ k\leqslant N;\quad \delta_{ik}=0\ \ (i\neq k),\quad \delta_{ik}=1\ \ (i=k) \tag{27}$$

Let us multiply (23) by $Y_{lm}(\omega)$ and integrate the result over $\omega\in\Omega$. Then, using the notation given above, we obtain the following vector equation:

$$\Phi_{m,n}^{\pm}=nB_mR_m^{\pm}\Phi_{m,n-1}^{\pm},\qquad n\geqslant 1 \tag{28}$$

The recurrence formula (28) enables us to express all vectors $\Phi_{m,n}^{\pm}(\eta)$ for $n\geq 1$, in terms of the vector $\Phi_{m,0}^{\pm}(\eta)$:

$$\Phi_{m,n}^{\pm}=n!\,(B_mR_m^{\pm})^n\Phi_{m,0}^{\pm},\qquad n\geqslant 1 \tag{29}$$

Let us now rewrite Formula (23) as follows

$$D^{n-2}F_n^{\pm}=n\lim_{\varepsilon\to+0}\frac{1}{\mu-\eta\mp i\varepsilon}\sum_{m=-N}^{N}\sum_{l=|m|}^{N}\frac{b_l}{2l+1}Y_{lm}(\omega)\|\Phi_{m,n-1}^{\pm}\|_l=$$

$$= n \lim_{\varepsilon \to +0} \frac{1}{\mu - \eta \mp i\varepsilon} \sum_{m=-N}^{N} B_m \mathbf{Y}_m(\omega) \cdot \mathbf{\Phi}_{m,n-1}^{\pm}, \qquad n \geqslant 1$$

This, together with (28), will yield

$$D^{n-2} F_n^{\pm} = n! \lim_{\varepsilon \to +0} \frac{1}{\mu - \eta \mp i\varepsilon} \sum_{m=-N}^{N} B_m \mathbf{Y}_m(\omega) \cdot (B_m R_m^{\pm}(\eta))^{n-1} \mathbf{\Phi}_{m,0}^{\pm}(\eta), \quad n \geqslant 1$$

To find $F_n^{\pm}$ from (30), we must determine their boundary properties for $n \geq 3$. From (19) it follows, that

$$\lim_{\sigma\eta \to \infty} \int_{\Omega} \gamma(\omega \cdot \omega') F_n^{\pm}(\eta, \omega') d\omega' = 0, \qquad n \geqslant 1 \tag{31}$$

Replacing in (10) F_n with $F_n^{\pm}$, differentiating the obtained relations s times with respect to η for each $n \geq 4$ ($s = 0, 1, \ldots, n-4$) and taking (19) and (31) into account, we obtain

$$\lim_{\sigma\eta \to \infty} \frac{\partial^s}{\partial \eta^s} F_n^{\pm} = 0, \qquad n \geqslant 3, \quad s = 0, 1, \ldots, n-3 \tag{32}$$

We shall show that the functions $\| R_m^{\pm}(\eta) \|_{ik}$ and $\| \mathbf{\Phi}_{m,0}^{\pm}(\eta) \|_l$ where the signs denote, respectively, the limit values from above and from below on the real axis of the complex variable η, can be analytically continued over the whole complex η-plane with a cut $[-1, 1]$. The definition (26) of the matrix $R_m^{\pm}(\eta)$ implies, that

$$\| R_m^{\pm} \|_{ik} = \lim_{\varepsilon \to +0} \int_{-1}^{1} \frac{P(\mu) d\mu}{\mu - \eta \mp i\varepsilon}, \qquad \operatorname{Im} \eta = 0$$

where $P(\mu)$ denotes a certain polynomial. From this we can infer, using the general properties of the Cauchy's integral, that $\| R_m^{\pm}(\eta) \|_{ik}$ taken as a function of the complex variable η is analytic for $\eta \overline{\in} [-1, 1]$; it can also be shown, that the points $\eta = \pm 1$ are its branch points. The matrix $R_m(\eta)$ given by

$$\| R_m(\eta) \|_{ik} = \int_{\Omega} \frac{\overline{Y}_{lm}(\omega') (Y_{km}(\omega')}{\mu' - \eta} d\omega', \qquad |m| \leqslant i, \ k \leqslant N$$

for an arbitrary complex η on the plane with a cut $[-1, 1]$, will represent a uniformized value of the matrices $R_m \pm (\eta)$ obtained as the limit values from above and from below on the real axis of the variable η.

Components of the vector $\Phi_{m0}^{\pm}(\eta)$ are found from Formulas (16) and (24)

$$\| \mathbf{\Phi}_{m,0}^{\pm}(\eta) \|_l = \frac{1}{2} \overline{Y}_{lm}(\omega_0) \lim_{\varepsilon \to +0} \frac{1}{\mu_0 - \eta \mp i\varepsilon}, \qquad \operatorname{Im} \eta = 0$$

and the uniformized value of the vectors $\Phi_{m,0}^{\pm}(\eta)$ for any $\eta \neq \mu_0$ is given in this case by

$$\| \mathbf{\Phi}_{m,0}(\eta) \|_l = \frac{1}{2} \overline{Y}_{lm}(\omega_0) \frac{1}{\mu_0 - \eta}$$

Since $\mu_0 \in [-1, 1]$, then $\| \Phi_{m,0}(\eta) \|_l$ is analytic for $\eta \overline{\in} [-1, 1]$.

Taking the analytic properties of $\| R_m(\eta) \|_{ik}$ and $\| \Phi_{m,0}(\eta) \|_l$ into account we find, that Eq. (30) with the boundary conditions given by (32) has, for Im $\eta = 0$, the following solution

$$F_n^{\pm} = n! \lim_{\lambda \to n-3} \frac{1}{\Gamma(\lambda + 1)} \int_{C_\sigma^{\pm}} \frac{(\eta - z)^{\lambda}\, dz}{\mu - z} \sum_{m=-N}^{N} B_m \mathbf{Y}_m(\omega) \cdot (B_m R_m(z))^{n-1} \mathbf{\Phi}_{m,0}(z), \quad n \geqslant 1 \tag{33}$$

where the integration around the contour $C_\sigma^{\pm}$ begins either at $z = \infty$ when $\sigma > 0$, or at $z = -\infty$ when $\sigma < 0$ and, depending on the sign of the function $F_n^{\pm}$ passes, respectively, along the line Im $z = \pm i\epsilon$ to reach the point $z = = \eta \pm i\varepsilon,\ \varepsilon > 0,\ |\varepsilon| \to 0$.

Validity of the formula (33) for $n \geq 3$ follows from the Cauchy formula for computing an n-tuple primitive function, while for $n = 1, 2$ it follows from the fact that when $\lambda \to n$ ($\eta = 1, 2, \ldots$), then the generalized function $x^{\lambda}\theta(x) / \Gamma(\lambda + 1)$ is equivalent to $\delta^{(n-1)}(x)$, while the generalized function $x^{\lambda}\theta(-x) / \Gamma(\lambda + 1)$ is equivalent to $(-1)^{(n-1)}\delta^{(n-1)}(x)$ (see e.g. [4]).

Formulas (9), (21) and (33) yield the particle distribution density $\psi(t, x, \omega)$ for the formal plane source (7)

$$\psi(t, x, \omega) = \psi(t, x, \mu, \mu_0, \varphi, \varphi_0) = \psi_0 + \psi_1 + \psi_2 + \tag{34}$$

$$+ \frac{h^3 e^{-\beta vt}}{\pi i} \int_C \frac{dz}{\mu - z} \sum_{m=-N}^{N} B_m \mathbf{Y}_m(\omega) \cdot (B_m R_m(z))^2 e^{hvt\,(\eta - z) B_m R_m(z)} \mathbf{\Phi}_{m,0}(z)$$

Here the contour of integration C begins and ends at the points $z = x/vt$, situated, respectively, on the lower and upper boundary of the cut $[-1, 1]$, passes the cut on the right when $\sigma > 0$ and on the left when $\sigma < 0$ (actually the cut may be passed on either side for any σ, since the expression under the integral sign in (34) is analytic outside the cut and its modulus decreases faster than $|z|^{-1}$ when $|z| \to \infty$). For the quantities ψ_0, ψ_1 and ψ_2 in (34) we have the following expressions:

$$\psi_0 = e^{-\beta vt} \delta^{(2)}(x - \mu vt)\, \delta(\mu - \mu_0)\, \delta(\varphi - \varphi_0) \tag{35}$$

$$\psi_1 = \lim_{\lambda \to -2} \frac{h e^{-\beta vt}}{\pi i (vt)^2 \Gamma(\lambda + 1)} \int_C \frac{dz\, (\eta - z)^{\lambda}}{\mu - z} \sum_{m=-N}^{N} B_m \mathbf{Y}_m(\omega) \cdot \mathbf{\Phi}_{m,0}(z) \tag{36}$$

$$\psi_2 = \lim_{\lambda \to -1} \frac{h^2 e^{-\beta vt}}{\pi i vt \Gamma(\lambda + 1)} \int_C \frac{dz\, (\eta - z)^{\lambda}}{\mu - z} \sum_{m=-N}^{N} B_m \mathbf{Y}_m(\omega) \cdot B_m R_m(z)\, \mathbf{\Phi}_{m,0}(z) \tag{37}$$

The required fundamental solution $I(t, \mathbf{x}, \omega)$ of a unsteady transport equation in solid geometry is now obtained by putting $\psi(t, x, \omega)$ given by (34) to (37), into (8).

In conclusion we note, that the given method can be applied, after slight formal modifications, to the problems of any spatial dimensionality. This statement is justified by the existence of a known expansion of the δ-function in terms of the plane waves, in the space of any dimensionality (see e.g. [4]).

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